Asian Journal of Control, Vol. 20, No. 1, pp. 577–584, January 2018
Published online 25 May 2017 in Wiley Online Library (wileyonlinelibrary.com) DOI: 10.1002/asjc.1546

–Brief Paper–

BIPARTITE LINEAR χ-CONSENSUS OF DOUBLE-INTEGRATOR MULTI-AGENT SYSTEMS WITH MEASUREMENT NOISE

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ABSTRACT

The bipartite consensus problem is investigated for double-integrator multi-agent systems in the presence of measurement noise. A distributed protocol with time-varying consensus gain is proposed. By using tools of state transition matrix and algebraic graph theory, necessary and sufficient conditions for the designed protocol to be a mean square bipartite linear χ-consensus protocol are given. It is shown that the signed digraph being structurally balanced and having a spanning tree are not only sufficient, but also necessary for bipartite consensus. Furthermore, the protocol is proved to be a mean square bipartite average consensus protocol if the signed graph is weight balanced.

Key Words: Bipartite linear χ-consensus, double-integrator, multi-agent systems, measurement noise.

I. INTRODUCTION

"Consensus" – to reach an agreement through collaboration– has been a hot topic for years in the distributed control of multi-agent systems (MASs) [1–4]. Within this general framework, it is realized that some agents may cooperate while others may compete [5], and therefore the concept of bipartite consensus is proposed [6]. For this type of consensus, the states of all agents converge to the same value except for their signs. Though still at the early stage, various works have been done in this field. To name a few, in [6], distributed Laplacian-like protocols are designed for first-order integrator MASs and bipartite consensus can be achieved if and only if the strongly connected signed digraph is structurally balanced; sufficient conditions for bipartite consensus are given for signed digraph with a spanning tree in [7]; the switching topology case is considered in [8]; the first-order MASs is investigated in [9], and it is shown that for general linear MASs, bipartite consensus over signed digraphs are equivalent to traditional consensus over nonnegative graphs, if the signed digraph is structurally balanced and has a spanning tree in [10].

On a parallel line to bipartite consensus, the effects of measurement noise have been fully considered for the conventional consensus problem. For example, in [11], a time-varying gain is introduced and then stochastic approximation protocols are designed for the discrete-time first-order integrator MASs, which is then extended in [12] to the continuous-time case, and more relevant works can be seen in [13,14]. Furthermore, in [15,16], measurement noise is modeled to be multiplicative. In [15], the noise intensities are assumed to be proportional to the absolute value of the relative states of an agent and its neighbors. For fixed and switching topology cases, sufficient conditions to achieve mean square consensus and strong consensus are given, respectively. Then, intensity function [15] is extended in [16] to the vector function.

We notice that the combination of bipartite consensus and measurement noise has not been taken good care of, and the only available ones are for first-order MASs under undirected signed graphs [17]. However, it is realized that various applications in the real world, such as the multirobot system [2] and the multivehicle system [18], are usually modeled to be of the second-order, and therefore there is the practical needs of investigating the effects of measurement noise on bipartite consensus for second-order MASs.

Following the above discussions, in this work we investigate bipartite consensus for double-integrator MASs with measurement noise under signed digraphs. Similar to [11–14,19], a time-varying consensus gain is introduced in the design of the bipartite consensus protocol. In the current problem setting, the row sum of Laplacian is nonzero and hence the Laplacian may be

Manuscript received September 19, 2016; revised January 9, 2017; accepted March 14, 2017.
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This work was supported in part by the National Natural Science Foundation of China (Nos. 61104136, 61673350), in part by Natural Science Foundation of Shandong Province (Grant Nos. ZR2010FQ002, ZR2016FQ09) and in part by the Scientific Research Foundation of Qufu Normal University.

a positive definite matrix, different from standard graph theory in conventional consensus. This thus fails conventional analysis tools in [14] and [19]. On the other hand, with measurement noise the closed-loop system is a time-varying stochastic differential equation, thus failing the analysis tools in [6–10] which are for bipartite consensus in the absence of measurement noise.

To overcome difficulties caused by measurement noise and hostile interactions, state transition matrix and algebraic graph theory are employed. The obtained conditions for the proposed protocol to be a mean square bipartite linear χ-consensus protocol are not only sufficient, but also necessary. We show that the signed digraph $\mathcal{G}$ to be structurally balanced and having a spanning tree are the weakest communication assumption for ensuring a mean square bipartite linear χ-consensus. Under the protocol, the positions/velocities of all agents converge in mean square to a random but the same value except the signs. Moreover, the mathematical expectation of the positions is a linear χ function of the initial positions and velocities of the agents, while the variance and mathematical expectation of velocities are both zero. Particularly, if $\mathcal{G}$ is also weight balanced, then the protocol is proved to be a mean square bipartite average consensus protocol.

The remainder of the paper is organized as follows. We introduce the necessary basic concepts on weighted signed graphs, and formulate the problem of interest in Section II. The main results are presented in Section III. Simulation examples are provided in Section IV, and Section V concludes the paper.

Notation. In this paper, $\mathbf{1}$ is the column vector with all ones. Re(λ) is the real part of λ. For given random variables x and y, Cov(x, y) denotes their covariance. $E(x)$ is the mathematical expectation of x and $D(x)$ is its variance.

II. PRELIMINARIES AND PROBLEM FORMULATION

2.1 Preliminaries on weighted signed graph

Multi-agent systems with hostile interactions are considered. The interaction topology of agents is represented by a weighted signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ where $\mathcal{V} = \{1, \cdots, N\}$ is the set of nodes and $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ is the set of edges. $(j, i) \in \mathcal{E}$ if and only if there is an information flow from j to i. The neighbor set of agent i is denoted by $\mathcal{N}_i = \{ j \in \mathcal{V} \mid (j, i) \in \mathcal{E}\}$. $\mathcal{A} = (\alpha_{ij}) \in R^{N\times N}$ is a weighted adjacency matrix. $\alpha_{ij} \neq 0$ if and only if $(j, i) \in \mathcal{E}$. $\alpha_{ij} < 0$ means competition and $\alpha_{ij} > 0$ means cooperation between agents i and j, respectively. In this paper, we always assume $\alpha_{ii} = 0$, $\alpha_{ij}\alpha_{ji} \geq 0$, $i, j = 1, \cdots, N$. $\mathcal{L} = \mathcal{D} - \mathcal{A}$ is called Laplacian of $\mathcal{G}$, where $\mathcal{D} = \text{diag}\left(\sum_{j=1}^{N} |\alpha_{1j}|, \cdots, \sum_{j=1}^{N} |\alpha_{Nj}|\right)$. A weighted signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is said weight balanced if $\sum_{j=1}^{N} |\alpha_{ij}| = \sum_{j=1}^{N} |\alpha_{ji}|$, $i = 1, \cdots, N$. A weighted signed digraph $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is said structurally balanced if there exist two subsets $\mathcal{V}_1$, $\mathcal{V}_2$, $\mathcal{V}_1 \bigcup \mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1 \bigcap \mathcal{V}_2 = \varnothing$, such that $\alpha_{ij} \geq 0$, $\forall i, j \in \mathcal{V}_l$, $(l \in \{1, 2\})$, $\alpha_{ij} \leq 0$, $\forall i \in \mathcal{V}_l, j \in \mathcal{V}_r$, $(l \neq r, l, r \in \{1, 2\})$. It is said structurally unbalanced otherwise.

Laplacian $\mathcal{L}$ has a close relationship with the connectivity of a weighted signed digraph $\mathcal{G}$, as stated below.

Lemma 1 ([20]). If $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is structurally balanced, then $\mathcal{L}$ has at least one zero eigenvalue and all the other eigenvalues are in the open right half plane. Particularly, $\mathcal{L}$ has exactly one zero eigenvalue if and only if $\mathcal{G}$ has a spanning tree.

2.2 Problem formulation and useful lemmas

Consider the double-integrator MASs with N agents, indexed from 1 to N. The dynamics of the ith agent is

$$\dot{x}_i(t) = v_i(t), \quad \dot{v}_i(t) = u_i(t), \quad i = 1, \cdots, N, \tag{1}$$

where $x_i(t) \in \mathbb{R}$, $v_i(t) \in \mathbb{R}$, $u_i(t) \in \mathbb{R}$ are the position, velocity and control input of the ith agent, respectively.

In view of the measurement noise, the ith agent is assumed to receive its neighbors' information $z_{x_{ij}}(t) = x_j(t)+b_{x_{ij}}\theta_{x_{ij}}(t)$, and $z_{v_{ij}}(t) = v_j(t)+b_{v_{ij}}\theta_{v_{ij}}(t), j \in \mathcal{N}_i$, where $z_{x_{ij}}(t)$, $z_{v_{ij}}(t)$ denote the measured position and velocity of agent j by agent i, respectively. $\{\theta_{x_{ij}}(t), \theta_{v_{ij}}(t),\ i, j = 1, \cdots, N\}$ are independent standard white noise and $\{b_{x_{ij}} > 0, b_{v_{ij}} > 0,\ i, j = 1, \cdots, N\}$ are the noise intensity.

The so-called bipartite linear χ-consensus problem under measurement noise is to design a distributed protocol for the MASs in (1) such that the positions/velocities of all agents converge to a random but the same value except their signs, while the mathematical expectation is a linear χ function of the initial states.

For the ith agent, we design the following protocol:

$$\begin{aligned} u_i(t) = -v_i(t) + k(t) \sum_{j\in\mathcal{N}_i} |\alpha_{ij}| \Big[\Big(& sgn(\alpha_{ij}) z_{x_{ij}}(t) \\ & -x_i(t)\Big) + \Big(sgn(\alpha_{ij}) z_{v_{ij}}(t) - v_i(t)\Big)\Big], \end{aligned} \tag{2}$$

where the piecewise continuous function $k(t) : [0, +\infty) \to (0, +\infty)$ is the time-varying gain.

Remark 1. Equation (2) is the protocol in [19] if the signed digraph $\mathcal{G}$ is traditional, *i.e.*, $\alpha_{ij} \geq 0, i, j = 1, \cdots, N$. (2) is a distributed protocol since only the states of agent i and its neighbors are used.

Define $X_i(t) = (x_i(t), v_i(t))^T$ and $X(t) = (X_1^T(t), \cdots, X_N^T(t))^T \in \mathbb{R}^{2N}$, respectively. Substituting the protocol in (2) to the system in (1) yields

$$\begin{aligned} \mathrm{d}X(t) =& [I_N \otimes \Delta + k(t)\mathcal{L} \otimes \Upsilon]X(t)\mathrm{d}t \\ &+ k(t)\Gamma \mathrm{d}W(t), \end{aligned} \tag{3}$$

where $\Delta = \begin{pmatrix} 0 & 1 \\ 0 & -1 \end{pmatrix}$, $\Upsilon = \begin{pmatrix} 0 & 0 \\ -1 & -1 \end{pmatrix}$, $\Gamma = \mathrm{diag}(\Gamma_1, \cdots, \Gamma_N) \in \mathbb{R}^{2N \times 2N^2}$, $\Gamma_i = \left(-\alpha_{i1}\begin{pmatrix} 0 & 0 \\ b_{x_{i1}} & b_{v_{i1}} \end{pmatrix}, \cdots, -\alpha_{iN}\begin{pmatrix} 0 & 0 \\ b_{x_{iN}} & b_{v_{iN}} \end{pmatrix}\right)$, $i = 1, \cdots, N$, $\int_0^t \theta_{x_{ij}}(s)\mathrm{d}s = W_{x_{ij}}(t)$, $\int_0^t \theta_{v_{ij}}(s)\mathrm{d}s = W_{v_{ij}}(t)$, $i, j = 1, \cdots, N$ and $W(t) = (W_{x_{11}}(t), W_{v_{11}}(t), \cdots\cdots, W_{x_{NN}}(t), W_{v_{NN}}(t))^T$ is a $2N^2$ dimensional standard Brownian motion.

The closed-loop system in (3) is a time-varying stochastic system. The following definition is useful to describe its asymptotic behaviour.

Definition 1. $\mathcal{U} = \{u_i, i = 1, \cdots, N\}$ is said to be a mean square bipartite linear χ-consensus protocol, if for any $X_i(0) \in \mathbb{R}^2$, $i = 1, \cdots, N$, there exist random variables x^* and v^* such that $\lim_{t\to\infty} E[x_i(t) - \varphi_i x^*]^2 = 0$, and $\lim_{t\to\infty} E[v_i(t) - \varphi_i v^*]^2 = 0$, where $E(x^*) = \chi(x_1(0), v_1(0), \cdots, x_N(0), v_N(0))$, $D(x^*) < \infty$, $E(v^*) = D(v^*) = 0$, $\chi(\cdot)$ is a linear function and $\varphi_i \in \{\pm 1\}(i = 1, \cdots, N)$ is independent of the initial states.

Remark 2. Definition 1 means that regardless of the initial states, the positions/velocities of all agents converge in mean square to a random but the same value except their signs. Definition 1 is not equivalent to the definition of mean square bipartite consensus protocol in [17].

Finally, the following assumptions are needed in the analysis that follows.

(A1) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ has a spanning tree.
(A2) $\mathcal{G} = (\mathcal{V}, \mathcal{E}, \mathcal{A})$ is structurally balanced.
(A3) $\int_0^\infty k(s)\mathrm{d}s = \infty$.
(A4) $\int_0^\infty k^2(s)\mathrm{d}s < \infty$.

Remark 3. (i) In the presence of measurement noise, from Theorem 1 of [17] we know that $\mathbf{A_1}$ and $\mathbf{A_2}$ are sufficient conditions on communication topology to ensure a mean square bipartite consensus for first-order integrator MASs. (ii) (A3) and (A4) are common assumptions in stochastic approximation theory. A possible option for $k(t)$ satisfying (A3) and (A4) has been given in [12].

For the closed-loop system in (3), the following lemma discusses the performance issue of its state transition matrix $\Psi(t, t_0)(t_0 \geq 0)$. It is instrumental in proving the main results in the following section.

Lemma 2. If the protocol in (2) is a mean square bipartite linear χ-consensus protocol, then there exists $\beta \in \mathbb{R}^{2N}$ such that $\lim_{t\to\infty} \Psi(t, 0) = f\beta^T$, where $f = (\varphi_1, 0, \cdots, \varphi_N, 0)^T \in \mathbb{R}^{2N}$.

Proof. According to Definition 1, for any given $X_i(0) \in \mathbb{R}^2$, $i = 1, \cdots, N$, there exist x^* and v^* such that $\lim_{t\to\infty} E[x_i(t) - \varphi_i x^*]^2 = 0$, and $\lim_{t\to\infty} E[v_i(t) - \varphi_i v^*]^2 = 0$, where $\varphi_i = \pm 1$, $i = 1, \cdots, N$, $E(x^*) = \chi(x_1(0), v_1(0), \cdots, x_N(0), v_N(0))$ and $E(v^*) = 0$. Denote $\varphi = (\varphi_1, \cdots, \varphi_N)^T$. Then $\lim_{t\to\infty} E\|X(t) - \varphi \otimes (x^*, v^*)^T\|^2 = 0$. From (3), one obtains $X(t) = \Psi(t, 0)X(0) + \int_0^t k(s)\Psi(t, s)\Gamma \mathrm{d}W(s)$. Without loss of generality, we assume that $\int_0^t k(s)\Psi(t, s)\Gamma \mathrm{d}W(s)$ converges to Y^* in mean square. Therefore,

$$\varphi \otimes (Ex^*, 0)^T = \lim_{t\to\infty} \Psi(t, 0)X(0) + EY^* \tag{4}$$

Note that $X(0)$ is arbitrary. Thus, for $2X(0)$ and $3X(0)$, there exist $\bar{\varphi}$, $\bar{x^*}$ and $\hat{\varphi}$, $\hat{x^*}$ such that

$$\begin{aligned} \bar{\varphi} \otimes (E\bar{x^*}, 0)^T &= 2\lim_{t\to\infty} \Psi(t, 0)X(0) + EY^*, \\ \hat{\varphi} \otimes (E\hat{x^*}, 0)^T &= 3\lim_{t\to\infty} \Psi(t, 0)X(0) + EY^*. \end{aligned} \tag{5}$$

By (4) and (5), one has

$$\varphi \otimes (Ex^*, 0)^T = \bar{\varphi} \otimes 2(E\bar{x^*}, 0)^T - \hat{\varphi} \otimes (E\hat{x^*}, 0)^T. \tag{6}$$

This implies that $\varphi = \pm\bar{\varphi} = \pm\hat{\varphi}$. If not, we assume $\varphi = \pm\bar{\varphi} = \pm\hat{\varphi}$ does not hold. Without loss of generality, we take $\varphi \neq \bar{\varphi} = \hat{\varphi}$ as an example. In this case, using (6) one obtains $Ex^* = 2E\bar{x^*} - E\hat{x^*}$ and $Ex^* = -2E\bar{x^*} + E\hat{x^*}$. Hence, $Ex^* = 0$. This contradicts the statement that (2) is a mean square bipartite linear χ-consensus protocol. Similarly, other cases can be proved.

Due to (5), we assume $\lim_{t\to\infty} \Psi(t, 0)X(0) = \varphi \otimes (Ed_x^*, 0)^T$. Since $X(0)$ is arbitrary, we can take $e_i = (0, \cdots, 0, \underbrace{1}_{i}, 0, \cdots, 0)^T (i = 1, \cdots, 2N)$ as $X(0)$, respectively. Then $\lim_{t\to\infty} \Psi(t, 0) = \left(\varphi \otimes (Ed_{e_1}^*, 0)^T, \cdots, \varphi \otimes (Ed_{e_{2N}}^*, 0)^T\right) = (\varphi_1, 0, \cdots, \varphi_N, 0)^T (Ed_{e_1}^*, \cdots, Ed_{e_{2N}}^*)_{1\times 2N} \triangleq f\beta^T$.

III. MAIN RESULTS

Theorem 1. For the system in (1), the protocol in (2) is a mean square bipartite linear χ-consensus protocol if and only if assumptions (A1)–(A4) hold.

Proof.
Sufficiency. Since (A1) and (A2) hold, Lemma 1 implies that Laplacian $\mathcal{L}$ has exactly one zero eigenvalue and all the other eigenvalues are in the open right half plane. Then there exists an invertible matrix $H \in \mathbb{C}^{N\times N}$ such that

$$H^{-1}\mathcal{L}H = \Lambda = \mathrm{diag}(0, Q_2, \cdots, Q_g), \tag{7}$$

where Q_i is a Jordan block corresponding to λ_i and $\mathrm{Re}(\lambda_i) > 0$, $i = 2, \cdots, g$. Combining this with Lemma 4 in [19], we obtain the state transition matrix $\Psi(t, t_0) = (H \otimes I_2)\mathrm{diag}\left(\Psi_{Q_1^0}(t, t_0), \Psi_{Q_2^{\lambda_2}}(t, t_0), \cdots\cdots, \Psi_{Q_g^{\lambda_g}}(t, t_0)\right)(H^{-1} \otimes I_2)$, where $\Psi_{Q_1^0}(t, t_0) = \begin{pmatrix} 1 & 1-e^{-t+t_0} \\ 0 & e^{-t+t_0} \end{pmatrix}$ and $\Psi_{Q_r^{\lambda_r}}(t, t_0)$, $r = 2, \cdots, g$ follows from Lemma 4 in [19]. Since (A3) holds, $\lim_{t\to\infty} \Psi_{Q_r^{\lambda_r}}(t, t_0) = \mathbf{0}$, $r = 2, \cdots, g$. Thus

$$\lim_{t\to\infty} \Psi(t, t_0) = (H \otimes I_2)\mathrm{diag}(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{0}, \quad \cdots, \mathbf{0}) \times(H^{-1} \otimes I_2). \tag{8}$$

Using Cauchy Criterion ([21]) and retracing the same arguments from Theorem 1 of [19], one obtains that $\int_0^t k(s)\Psi(t,s)\Gamma \mathrm{d}W(s)$ converges to a random vector X_2^* in mean square. Therefore, $X(t) = \Psi(t, 0)X(0) + \int_0^t k(s)\Psi(t,s)\Gamma \mathrm{d}W(s)$ converges in mean square to a random vector $X^* = (x_1^*, v_1^*, \cdots, x_N^*, v_N^*)^T$. By (8), one has $E(X^*) = (H \otimes I_2)\mathrm{diag}(\begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}, \mathbf{0}, \cdots, \mathbf{0})(H^{-1} \otimes I_2)X(0)$. Let the first column of H be h_r and the first row of H^{-1} be $h_l^T = (l_1, \cdots, l_N)$. Then $h_l^T h_r = 1$. By (7), one obtains $\mathcal{L}H = H\Lambda$ and $H^{-1}\mathcal{L} = \Lambda H^{-1}$. Thus $h_l^T\mathcal{L} = \mathbf{0}$ and $\mathcal{L}h_r = \mathbf{0}$, *i.e.*, h_l and h_r are the left and right eigenvectors associated with zero eigenvalue of $\mathcal{L}$, respectively. From (A2) and Lemma 1 in [6], there exists $D = \mathrm{diag}(d_1, \cdots, d_N)(d_i \in \{\pm 1\})$ such that $D\mathcal{A}D$ has all nonnegative elements. Hence $D\mathcal{L}D\mathbf{1} = \mathbf{0}$. Since $\mathcal{L}$ has only one zero eigenvalue, the eigenspace associated with the zero eigenvalue is 1 dimensional. Without loss of generality we may assume that $h_r = D\mathbf{1} = (d_1, \cdots, d_N)^T$. Therefore, $E(x_i^*) = d_i \sum_{j=1}^N l_j[x_j(0) + v_j(0)]$, and $E(v_i^*) = 0, i = 1, \cdots, N$. Using the similar sufficiency proof of Theorem 1 in [19], one immediately obtains $D(X^*) = \left(\Omega_{ij}\right)_{N\times N} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$, where $\Omega_{ij} = d_i d_j \Omega^*$, and $\Omega^* = \int_0^\infty k^2(s)\mathrm{d}s \sum_{i=1}^N \sum_{j=1}^N \alpha_{ij}^2 l_j^2 (b_{x_{ij}}^2 + b_{v_{ij}}^2)$. Hence, $D(x_i^*) = \Omega^* < \infty$, $D(v_i^*) = 0$, $\mathrm{Cov}(x_i^*, x_j^*) = \Omega_{ij}$, and $\mathrm{Cov}(v_i^*, v_j^*) = 0$, $\forall i, j = 1, \cdots, N$. Therefore, for $\forall i = 1, \cdots, N$, $\lim_{t\to\infty} E[x_i(t) - d_i d_1 x_1^*]^2 \le \lim_{t\to\infty} E[x_i(t) - x_i^*]^2 + 2\lim_{t\to\infty}[E(x_i(t) - x_i^*)^2]^{\frac{1}{2}}[E(x_i^* - d_i d_1 x_1^*)^2]^{\frac{1}{2}} + E[x_i^* - d_i d_1 x_1^*]^2 = E[x_i^* - d_i d_1 x_1^*]^2 = 0$. Similarly, $\lim_{t\to\infty} E[v_i(t) - d_i d_1 v_1^*]^2 = 0$. Denote $\varphi_i = d_i d_1$, $i = 1, \cdots, N$. Then by Definition 1, (2) is a mean square bipartite linear χ-consensus protocol.

Necessity. We prove the necessity in the following five parts.

I. The necessity of (A$_3$). For simplicity, we consider an MASs of 3 agents, where $\mathcal{A} = (\alpha_{ij}) \in \mathbb{R}^{3\times 3}$ and $\alpha_{12} = \alpha_{21} = a > 0$, $\alpha_{13} = \alpha_{31} = -a$, $\alpha_{11} = \alpha_{22} = \alpha_{33} = \alpha_{23} = \alpha_{32} = 0$. Without loss of generality one may assume $a = 1$. Obviously, the eigenvalues of $\mathcal{L}$ are 0, 1 and 3. Suppose δ_0, δ_1 and δ_3 are unit right eigenvectors associated with 0, 1 and 3, respectively. Let $\Theta = (\delta_0, \delta_1, \delta_3)$. Then applying the protocol in (2), one obtains the state transition matrix $\Psi(t, 0) = (\Theta \otimes I_2)\mathrm{diag}(\Psi_0, \Psi_1, \Psi_3)(\Theta^{-1} \otimes I_2)$ of the closed-loop system, where $\Psi_0 = \begin{pmatrix} 1 & 1-e^{-t} \\ 0 & e^{-t} \end{pmatrix}$, $\Psi_1 = \begin{pmatrix} e^{-t} + n_1(t) & n_1(t) \\ n_3(t) - e^{-t} - n_1(t) & n_3(t) - n_1(t) \end{pmatrix}$, $\Psi_3 = \begin{pmatrix} e^{-t} + n_2(t) & n_2(t) \\ n_3(t) - e^{-t} - n_2(t) & n_3(t) - n_2(t) \end{pmatrix}$, $n_1(t) = \int_0^t e^{-t+s}e^{-\int_0^s k(\tau)\mathrm{d}\tau}\mathrm{d}s$, $n_2(t) = \int_0^t e^{-t+s}e^{-3\int_0^s k(\tau)\mathrm{d}\tau}\mathrm{d}s$ and $n_3(t) = e^{-\int_0^t k(s)\mathrm{d}s}$. If $\int_0^\infty k(s)\mathrm{d}s = \infty$ does not hold, then $\mathrm{rank}\left(\lim_{t\to\infty} \Psi(t, 0)\right) = 3$. Lemma 2 implies $\mathrm{rank}\left(\lim_{t\to\infty} \Psi(t, 0)\right) = \mathrm{rank}\left(f\beta^T\right) \le 1$. We arrive at a contradiction. Thus, $\int_0^\infty k(s)\mathrm{d}s = \infty$, *i.e.*, (A3) holds.
II. Prove $\mathcal{L}$ has exactly one zero eigenvalue. Firstly, we prove 0 is an eigenvalue of $\mathcal{L}$. By contradiction, we assume 0 is not an eigenvalue of $\mathcal{L}$. Then, from $\int_0^\infty k(s)\mathrm{d}s = \infty$, one immediately obtains that $\lim_{t\to\infty} \Psi(t, 0) = \mathbf{0}$. This together with (4) leads to $EY^* = \varphi \otimes (Ex^*, 0)^T$. Since EY^* is independent of $X(0)$, one obtains Ex^* is independent of $X(0)$. This contradicts the statement that $Ex^* = \chi(x_1(0), v_1(0), \cdots, x_N(0), v_N(0))$. Therefore, 0 is an eigenvalue of $\mathcal{L}$.

Suppose Q_1^0 is a Jordan block associated with 0. Then its dimension is 1. If not, without loss of generality one may assume Q_1^0 is 2 dimensional Jordan block associated with 0. A straightforward calculation shows that $\lim_{t\to\infty}\Psi_{Q_1^0}(t,0)$ does not exist, where $\Psi_{Q_1^0}(t,0)$ is a block of the state transition matrix $\Psi(t,0)$. Hence, $\lim_{t\to\infty}\Psi(t,0)$ does not exist. This contradicts Lemma 2. Other cases can be proved similarly. Therefore, the Jordan block associated with 0 is 1 dimensional.

Secondly, we prove 0 is an eigenvalue of multiplicity 1. Suppose the multiplicity of 0 is l. If $l > 1$, then without loss of generality we may assume $l = 2$. Due to the fact that the Jordan block associated with 0 is 1 dimensional, by a slight abuse of notation, we use (8) and get $\lim_{t\to\infty}\Psi(t,0) = (H\otimes I_2)\mathrm{diag}(\begin{pmatrix}1&1\\0&0\end{pmatrix},\begin{pmatrix}1&1\\0&0\end{pmatrix},\mathbf{0},\cdots,\mathbf{0})(H^{-1}\otimes I_2)$. Hence $\mathrm{rank}\left(\lim_{t\to\infty}\Psi(t,0)\right) = 2$. By Lemma 2, $\mathrm{rank}\left(\lim_{t\to\infty}\Psi(t,0)\right) = \mathrm{rank}(f\beta^T) \le 1$. This is a contradiction. Thus $l = 1$.

Finally, we know that $\mathcal{L}$ has exactly one zero eigenvalue.

III. The necessity of $(\mathbf{A_2})$**.** From **II**, $\mathcal{L}$ has exactly one zero eigenvalue. Thanks to (A3), one has (8). This together with Lemma 2 leads to $(H\otimes I_2)\mathrm{diag}(\begin{pmatrix}1&1\\0&0\end{pmatrix},\mathbf{0},\cdots,\mathbf{0})(H^{-1}\otimes I_2) = f\beta^T$. Note that the first column of H is h_r. The above equation implies $h_r = (\varphi_1,\cdots,\varphi_N)^T\cdot k^*$, where $k^* = \beta^T(h_r\otimes\begin{pmatrix}1\\0\end{pmatrix})$. Since $\mathcal{L}h_r = \mathbf{0}$, $\mathcal{L}(\varphi_1,\cdots,\varphi_N)^T = \mathbf{0}$. It follows that for $\forall j$, $\varphi_i\sum_{j\neq i}|\alpha_{ij}| = \sum_{j\neq i}\varphi_j\alpha_{ij}$, $j = 1,\cdots,N$. Recall $\varphi_i = \pm 1$, $\varphi_i^2 = 1$, $i = 1,\cdots,N$. There thus holds $\sum_{j\neq i}|\alpha_{ij}| = \sum_{j\neq i}\varphi_i\varphi_j\alpha_{ij}$. Therefore, $\varphi_i\varphi_j\alpha_{ij} = |\alpha_{ij}| \ge 0$. Let $\mathcal{V}_1 = \{i|\varphi_i = 1, i = 1,\cdots,N\}$ and $\mathcal{V}_2 = \{i|\varphi_i = -1, i = 1,\cdots,N\}$. Then $\mathcal{V}_1\bigcup\mathcal{V}_2 = \mathcal{V}$, $\mathcal{V}_1\bigcap\mathcal{V}_2 = \varnothing$ and $\alpha_{ij} \ge 0$ for $i,j\in\mathcal{V}_q, q\in\{1,2\}$, $\alpha_{ij}\le 0$ for $i\in\mathcal{V}_p, j\in\mathcal{V}_q, p\neq q\in\{1,2\}$. By definition, $\mathcal{G}$ is structurally balanced.

IV. The necessity of $(\mathbf{A_1})$**.** From **II, III** and Lemma 1, one knows that $\mathcal{G}$ has a spanning tree, *i.e.*, (A1) holds.

V. The necessity of $(\mathbf{A_4})$**.** Notice that $h_l^T = (l_1,\cdots,l_N)$ is the first row of H^{-1} and $h_l^T\mathcal{L} = \mathbf{0}$. Therefore, by (3), one has $\mathrm{d}\left(\left(h_l^T\otimes(1,1)\right)X(t)\right) = k(t)\left(h_l^T\otimes(1,1)\right)\Gamma\mathrm{d}W(t)$. Let $\xi^T = h_l^T\otimes(1,1)$. Then $\xi^T X(t) = \xi^T X(0) + \xi^T\Gamma\int_0^t k(s)\mathrm{d}W(s)$. From Definition 1, one derives that state $X(t)$ of the closed-loop system in (3) converges in mean square to a random vector with finite variance. Hence $\xi^T\Gamma\int_0^t k(s)\mathrm{d}W(s)$ converges in mean square to a random variable with finite variance. Therefore, $\lim_{t\to\infty}E[\xi^T\Gamma\int_0^t k(s)\mathrm{d}W(s)]^2 < \infty$. If (A4) does not hold, *i.e.*, $\int_0^\infty k^2(s)\mathrm{d}s = \infty$, then $\lim_{t\to\infty}E[\xi^T\Gamma\int_0^t k(s)\mathrm{d}W(s)]^2 = \lim_{t\to\infty}\xi^T\Gamma\Gamma^T\xi\int_0^t k^2(s)\mathrm{d}s = \infty$. This leads to a contradiction. Thus $\int_0^\infty k^2(s)\mathrm{d}s < \infty$.

Remark 4. Theorem 1 shows that in the presence of measurement noise, (A1)–(A4) are necessary and sufficient conditions to ensure a mean square bipartite linear χ-consensus protocol. Particularly, (A1) and (A2) are the weakest conditions on communication topology, while (A3) and (A4) are to ensure the convergence of the closed-loop system.

Remark 5. From the sufficiency proof we know that $\varphi_i = d_i d_1$, $i = 1,\cdots,N$, where d_i is the component of the right eigenvector associated with eigenvalue 0 of $\mathcal{L}$. Clearly, it is determined by the communication topology and independent of the initial state $X(0)$.

Remark 6. Under the assumption that $\mathcal{G}$ is structurally balanced and has a spanning tree, bipartite consensus problem is a traditional consensus problem. This is also true in the presence of measurement noise ([17]). Therefore, part of the sufficiency proof of Theorem 1 can be similarly derived from [19]. However, for the necessity proof, it is worth noting that conditions to be structurally balanced and having a spanning tree exist no longer as a premise, but as conclusions to be proved. So gauge transformation is invalid and the arguments in [19] are not applicable.

From the sufficiency proof of Theorem 1, it can be seen that $E(x^*) = \sum_{j=1}^N l_j(x_j(0) + v_j(0))$. It is a linear function of initial positions and velocities. Specially, in Theorem 1, if $l_j = \frac{d_j}{N}$, then the protocol in (2) is called a mean square bipartite average consensus protocol.

By Theorem 1, one immediately obtains the following result.

Corollary 1. For the system in (1), the protocol in (2) is a mean square bipartite average consensus protocol if and only if $\mathcal{G}$ is weight balanced and (A1)–(A4) hold.

Remark 7. Corollary 1 is consistent with Theorem 1 in [19] if all $\alpha_{ij}\ge 0$ in $\mathcal{G}$, $i,j = 1,\cdots,N$, *i.e.*, $\mathcal{G}$ is a traditional digraph.

IV. SIMULATION EXAMPLES

Example 1. Consider an MAS in (1) with five agents. The communication topology among them is represented by the signed digraph $\mathcal{G}_1 = (\mathcal{V},\mathcal{E},\mathcal{A})$, where $\mathcal{V}=\{1,2,3,4,5\}$. From Fig. 1 we know that $\mathcal{G}_1$ is structurally balanced and has a spanning tree.

Considering the effects of measurement noise, we take time-varying gain $k(t) = \frac{1}{t+1}, t \geq 0$ in protocol (2). Clearly, $k(t)$ satisfies (A3) and (A4). Suppose that the initial states of agents are $X_1(0) = (-4\ 0)^T$, $X_2(0) = (4\ 1)^T$, $X_3(0) = (2\ 2)^T$, $X_4(0) = (1\ -1)^T$ and $X_5(0) = (3\ -2)^T$, respectively. The evolution of positions/velocities of agents can be seen in Fig. 2 and Fig. 3, respectively. A simple calculation shows that the left eigenvector associated with eigenvalue 0 of $\mathcal{L}$ in $\mathcal{G}_1$ is $h_l = (l_1, \cdots, l_5)^T = (0\ 0\ 0\ 0\ 1)^T$. From Theorem 1 and Definition 1 we know that positions and velocities of agents converge in mean square to random variables $\pm x^*$ and $\pm v^*$, respectively. Moreover, $E(x^*) = \sum_{j=1}^{5} l_j[x_j(0) + v_j(0)] = 1$ and $E(v^*) = 0$. They are validated by Fig. 2 and Fig. 3, respectively.

Example 2. If the communication topology in Example 1 is represented by $\mathcal{G}_2$, then from Fig. 4 we know $\mathcal{G}_2$ not only satisfies (A1)–(A2) but also is weight balanced.

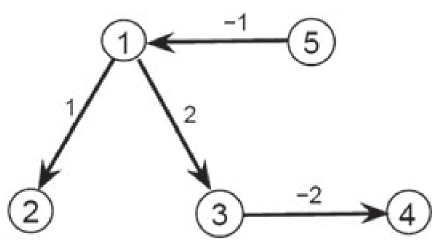

Fig. 1. Signed digraph $\mathcal{G}_1$.

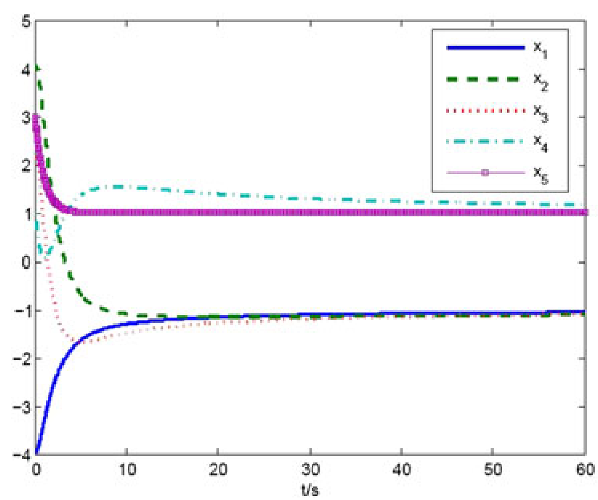

Fig. 2. Evolution of positions of agents. [Color figure can be viewed at wileyonlinelibrary.com]

To reduce the detrimental effects of measurement noise, a time-varing gain $k(t) = \frac{t}{t^2+1}$ which satisfies (A3) and (A4) is chosen in (2). The initial states are $X_1(0) = (5\ -1)^T$, $X_2(0) = (3\ 1)^T$, $X_3(0) = (-1\ -2)^T$, $X_4(0) = (-3\ 0)^T$ and $X_5(0) = (2\ -4)^T$, respectively. From Corollary 1 we know that the positions/velocities of agents 1, 2, 3 converge in mean square to random variable $x^*/\ v^*$, while agents 4 and 5's positions/velocities converge in mean square to $-x^*/-v^*$. Furthermore, $E(x^*) = \frac{1}{5}\sum_{j=1}^{5} \varphi_j[x_j(0) + v_j(0)] = 2$, $E(v^*) = 0$. Fig. 5 and Fig. 6 show that bipartite average consensus is achieved in the presence of measurement noise.

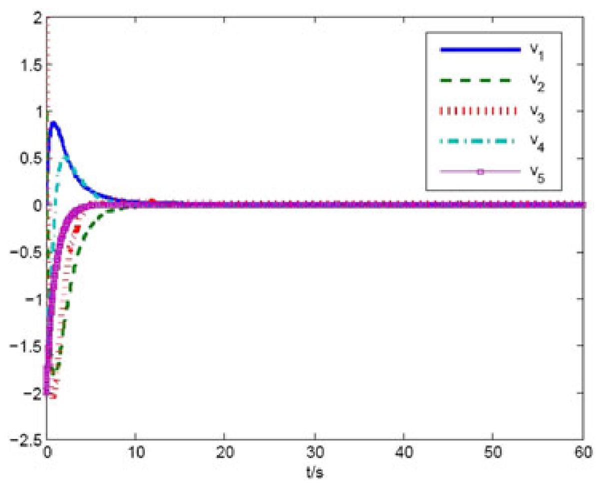

Fig. 3. Evolution of velocities of agents. [Color figure can be viewed at wileyonlinelibrary.com]

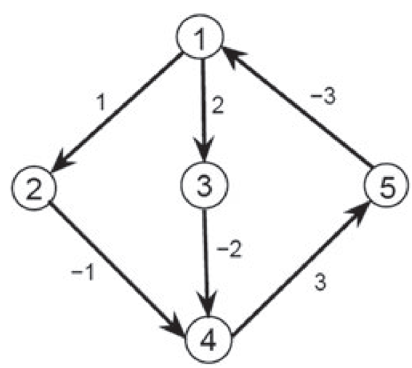

Fig. 4. Signed digraph $\mathcal{G}_2$.

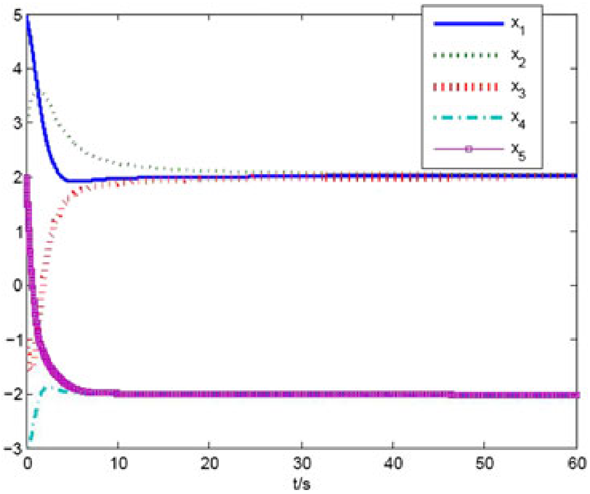

Fig. 5. Position trajectories in bipartite average consensus case. [Color figure can be viewed at wileyonlinelibrary.com]

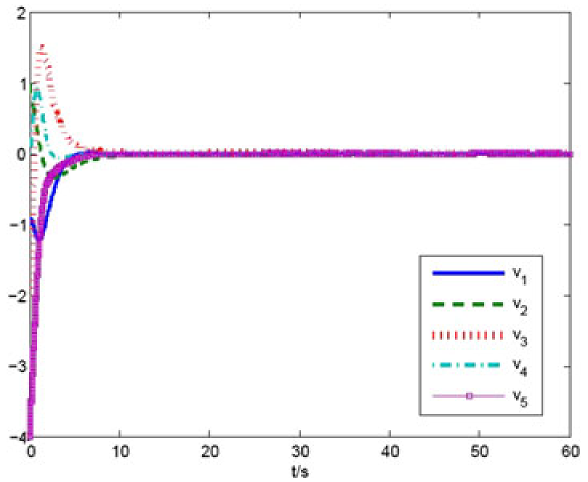

Fig. 6. Velocity trajectories in bipartite average consensus case. [Color figure can be viewed at wileyonlinelibrary.com]

V. CONCLUSIONS

The effects of measurement noise on bipartite consensus for the double-integrator MASs are investigated. By using the time-varying consensus gain and symbolic function, a distributed bipartite consensus protocol is designed for double-integrator MASs in the presence of measurement noise. Necessary and sufficient conditions for ensuring a mean square bipartite linear χ-consensus protocol are given. The obtained results are for double-integrator MASs under fixed topology, and the switching topology case will be our future focus.

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